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RELATION BETWEEN THE VERTICES OF THE PERMUTATION POLYTOPE AND ELEMENTS OF THE SYMMETRIC GROUP

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ABSTRACT

A permutation polytope is the convex hull of permutation matrices. An example of permutation polytope is the Birkhoff polytope which is all doubly stochastic matrices, this polytope is defined as the convex hull of all permutation matrices.The aim of this paper is to prove an open conjecture about Birkhoff polytope and symmetric group, we want to prove that there a one to-one map from the verticesof the Birkhoff polytope and the symmetric group S_n.

KEYWORDS: Birkhoff Polytope, Permutation Polytope, Symmetric Group

INTRODUCYION

Permutation polytope is one kind of polytopes. It is a polytope associated to a permutation group. Previously permutation polytope called permutahedra or permutahedron. The permutahedron was first written about by Schoute in 1911, [24]. In 1972 the phrase "permutation polytope" was used for the first time, [8]. It is naturally appeared in various contexts such as enumerative combinatorics, optimization and statistics, [1].

A number of authors have studied special classes of permutation polytopes. In 1977 Brualdi studied polytope associated permutation matrices to determine the dimension, for this polytope is $(n-1)^2+1$, [9]. In 1978 Young studied easy characterization giving neighbors on permutation polytope, [23]. In 1991 Brualdi and Liu computed the invariants of the permutation polytope of the Alternating group, [10]. In 1993 Onn and Thompson entered new concepts with permutation polytope, [22], [18]. In 2004 Hood and Perkinson described exponentially many facets of permutation polytope(solve conjecture in[10]), [15]. Also in the same year Hwang and Rothblum defined a permutation polytope which was the convex hull of the vectors corresponding to all permutations, [16]. In 2006 Guralnick and Perkinson investigated general permutation polytopes, their dimension and their graph, [14]. In 2009 Baumeister, Haase, Nill and Paffenholze studied a notion of equivalences, provided product theorem and discussion centrally symmetric permutation polytopes. They provided a number of combinatorial properties of permutation polytopes, [1]. In 2011 Burggraph obtained permutation polytope of permutation group and computed the volume of permutation polytope, [11]. In 2012 Baumeister, Haase, Nill and Paffenholze investigated the combinatorics and geometry of permutation polytopes associated to cyclic permutation groups i.e the convex hull of cyclic groups of permutation matrices, [2]. In 2013 Burggraf, Loera and omer computed volumes of permutation polytope associated to cyclic group, dihedral group and Frobenius group, [12]. In 2014 Baumeister, Haase, Nill and Paffenholze studied permutation polytope which obtained of dihedral group, [3].

In 2015 Barbara and Matthias clarified the notion of effectively equivalence and solve question showing that the effectively permutation group did not correspond to affinely equivalent polytopes this question was in paper of 2009 on permutation polytope, [4].

1 Basic Concepts

In this section some basic definitions and examples of permutation polytopeare given to consolidate our results.

Definition(1),[21]

All matrices obtained from n×n identity matrix by permuting the rows according to some permutation of the number 1 to n is called permutation matrices and denoted by M(G). So the number of permutation matrices is n!.

Example(1)

An example of permutation matrices is given by:

Let n=2 with identity matrix equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the permutation matrices are two, which are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Definition(2),[11]

Let A be a finite set {1,2,…,n}, a permutation group is defined as a group, whose element are permutations of A.

Definition(3),[1]

A permutation polytope is defined as the convex hull of a group or subgroup of permutation matrices.

Definition(4),[13]

Symmetric group is defined to be a group of all permutations of A where A is the finite set {1,2,...,n}, the group is said to be a symmetric group on n and denoted by S_n, Which has n! elements.

Definition(5)

Let g be one element of the permutation group G and A any matrix in the set of permutation matrices then,

$$\mathbf{g}=F(\mathbf{A})=\begin{cases} i & if \quad a_{ij}=1 \; and \; i=j \\ j & if \quad a_{ij}=1 \; and \; i\neq j \\ no\ solutions\ if\ a_{ij}=0 \end{cases}$$

Example(2)

To convert permutation matrices to a permutation group, we use the following steps:

First convert permutation polytope to a permutation group

Where permutation polytope p(G)=conv(M(G)) and

$$M(G)=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\}$$

The second step is obtained by taking any matrix A_1 from a permutation group and let g be any element of the permutation group, then:

$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3) = e$

$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$

$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$

In a similar way, it is computed

$g_4 = (12)(3)$, g_5=(13)(2)and $g_6 = (1)(23)$

Permutation group={e,(123),(132),(12),(13),(23)}

Definition(6),[5],[17]

The Birkhoff polytope B_n is the convex polytope consists of all doubly stochastic n×n matrices, and it is defined as:

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in R^{n^2} : \begin{matrix} \sum_j x_{jk} = 1 \, for \, all \, 1 \le k \le n \\ \sum_k x_{jk} = 1 \, for \, all \, 1 \le j \le n \end{matrix} \right\}$$

Remark(1)

1- The Birkhoff polytopeB_n= $P(S_n)$=conv($M(S_n)$),[2].

2-The Birkhoff polytopeB_n has dimension $(n-1)^2$, [19].

3-The polytope associate symmetric group have same dimension of the Birkhoff polytope $B_n$which is $(n-1)^2$.

4-The Birkhoff polytopeB_n has n^2 facets and n! vertices, [20].

Example(3)

H-representation of the Birkhoff polytope is given by the following:

Let n=2

$B_2 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ such that

$x_{11} + x_{12} = 1$

$x_{21} + x_{22} = 1$

$x_{11} + x_{21} = 1$

$x_{12} + x_{22} = 1$

This mean that $\begin{bmatrix} 1 & 10 & 0 \\ 0 & 01 & 1 \\ 1 & 01 & 0 \\ 0 & 10 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

A= $\begin{bmatrix} 1 & 10 & 0 \\ 0 & 01 & 1 \\ 1 & 01 & 0 \\ 0 & 10 & 1 \end{bmatrix}$, X= $\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$, b= $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ such that AX=b.

Example(4)

V-representation of Birkhoff polytope is given as follows, let n=3, the permutation matrices are:

$$M(G)=\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

B_3=conv(M(G)) such that every matrix in M(G) is a vertex of the Birkhoff polytope.

2 PROPERTIES ON PERMUTATION POLYTOPE

In this section a proof of an open conjecture for the relation between permutation polytope and symmetric group S_n is given by theorem(1) below, this open conjecture is given as an open problem in [6].

Theorem(1)

The vertices of the Birkhoff-von Neumann polytope are in one to-one correspondence with the elements of the symmetric group S_n.

Proof

Let f represent the function that make a correspond between the vertices of Birkhoff polytope and the permutation matrices and let g represent the function that make a correspond between the permutation matrices and the elements of the symmetric group S_n, we want to prove that fog is one to-one between the vertices of Birkoff polytope and the element of the symmetric group S_n.

By Birkhoff, proved that B_n is a convex polytope whose extreme points are the n! permutation matrices,[7]. This mean that every permutation matrix is represent by a vertex for Birkhoff polytope and the vertices of Birkhoff polytope and the permutation matrices is one to one.

Also there is a relation between symmtric group S_n and a permutation matrices, [24] using definition(5) which is given in this paper, we get the result

That is, let $x_1,x_2\in$ set of vertices of Birkhoff polytope, $x_1\neq x_2$

$g(x_1) \neq g(x_2)$

The function g is one to one, since the functions f and g are one to one thengof is one to one.

Therefore the vertices of the Birkhoff polytope are one to-one correspond with the elements of the symmetric group S_n.

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